A BOUNDED NORMAL LIGHT INTERIOR FUNCTION THAT POSSESSES NO POINT ASYMPTOTIC VALUES

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ABSTRACT

Bagemihl and Seidel have shown that the set of Fatou points of a normal holomorphic function in D is everywhere dense on C. We present an example of a bounded normal light interior function that possesses no point asymptotic values.

Let D be the unit disk, C the unit circle. Let f be a light interior function from D into the Riemann sphere W, i.e., let f be a continuous open map that does not take any continuum into a single point. It is know that f has a factorization $f = g \circ h$ where h is a homeomorphism of the unit disk onto either the unit disk or the finite complex plane and g is a non-constant meromorphic function [3]. We will be concerned with the case when the range of h is the unit disk.

We say that f has the *point asymptotic value c at* $e^{i\theta}$ if there exists a Jordan arc lying in D except for one end point $e^{i\theta}$ on which f has the limit c. The function f is *normal* if it is uniformly continuous with respect to the non-Euclidean hyperbolic metric ρ in D and the chordal metric in W [4]. Let h be a homeomorphism of D onto D. If h is uniformly continuous with respect to the non-Euclidean hyperbolic metric in both its domain and range, then we say that h is HUC. Since the composition of two uniformly continuous functions is uniformly continuous, the following theorem is immediate [5].

THEOREM A. *Let h be a homeomorphism of D onto D which is HUC. If g is a non-constant normal meromorphic function in D then the light interior function $f = g \circ h$ is normal.*

Fatou's theorem states that a bounded holomorphic function possesses radial limits at almost every point of C. The following result shows that a bounded normal light interior function need not posses any point asymptotic values.

Received August 15, 1969 and in revised form November 23, 1969.

THEOREM. *There exists a homeomorphism h of D onto D with the property: If g is a non-constant normal meromorphic function in D, then the light interior function $f = g \circ h$ is normal and possesses no point asymptotic values.*

Since a bounded holomorphic function is normal, we obtain the following corollary.

COROLLARY. *There exists a homeomorphism h of D onto D with the property: If g is a non-constant bounded holomorphic function in D, then the bounded light interior function $f = g \circ h$ is normal and possesses no point asymptotic values.*

Before proving the Theorem we establish the following lemma.

LEMMA. *There exists a homeomorphism h of D onto D such that the radii of D are mapped onto spirals and h is HUC.*

Proof. Let $\{R_n\}$ be a strictly increasing sequence of non-negative real numbers with $R_0 = 0$ for which $\rho(R_n, R_{n+1}) = 1/(1 - R_n^2)$. Let $\Phi(r)$ be a mapping of the interval $[0,1)$ onto itself defined by $\Phi(r) = (rR_1)/R_2$ for $0 \leqq r < R_2$ and satisfying the equation

$$\rho(R_{n-1}, \Phi(r))/\rho(R_{n-1}, R_n) = \rho(R_n, r)/\rho(R_n, R_{n+1})$$

for $R_n \leqq r < R_{n+1}$ $(n = 2, 3, \cdots)$. A straightforward calculation shows that if $R_n \leqq r_1 \leqq r_2 < R_{n+1}$, then $\rho(\Phi(r_1), \Phi(r_2)) \leqq \rho(r_1, r_2)$. Define a function $\Psi(r)$ on $[0,1)$ by $\Psi(r) = 2\pi\rho(0,r)/\rho(0,R_2)$ for $0 \leqq r < R_2$ and satisfying the equation $\Psi(r) = 2\pi\rho(R_n, r)/\rho(R_n, R_{n+1})$ for $R_n \leqq r < R_{n+1}$ $(n = 2, 3, \cdots)$.

Let the mapping h in D be defined by

$$h(z) = h(re^{i\theta}) = \Phi(r)\exp(i\theta + i\Psi(r)).$$

It is easy to verify that h is a homeomorphism of D onto D and that the radii of D are mapped onto spirals.

Set $A_n = \{z : R_n \leqq |z| < R_{n+1}\}$. Let $n \geqq 2$ be fixed but arbitrary and let z, $z' \in A_n$ with $\rho(z, z') < 1$; the proof will be complete if we can find a constant K independent of n, for which $\rho(h(z), h(z')) \leqq K\rho(z, z')$. We may assume that $z = re^{i\alpha}$ and $z' = r'e^{i\beta}$ with $r \leqq r'$. Then we have the following inequality

$$\begin{aligned} \rho(h(z), h(z')) \leqq\ & \rho(\Phi(r)\exp(i\alpha + i\Psi(r)), \Phi(r)\exp(i\beta + i\Psi(r))) \\ & + \rho(\Phi(r)\exp(i\beta + i\Psi(r)), \Phi(r)\exp(i\beta + i\Psi(r'))) \\ & + \rho(\Phi(r)\exp(i\beta + i\Psi(r')), \Phi(r')\exp(i\beta + i\Psi(r'))). \end{aligned}$$

From the fact that $\Phi(r) \leqq r$ we obtain

$$\rho(\Phi(r)\exp(i\alpha + i\Psi(r)), \Phi(r)\exp(i\beta + i\Psi(r)))$$
$$= \rho(\Phi(r)e^{i\alpha}, {}_{\phi}(r)\, e^{i\beta}) \leqq \rho(re^{i\alpha}, re^{i\beta}) \leqq \rho(z, z').$$

From the facts that $\Phi(r) \leqq R_n$ and $\rho(R_n, R_{n+1}) = 1/(1 - R_n^2)$ and [2, §43], we obtain

$$\rho(\Phi(r)\exp(i\beta + i\Psi(r)), \Phi(r)\exp(i\beta + i\Psi(r')))$$
$$\leqq \int_{\Psi(r)}^{\Psi(r')} \frac{\Phi(r)d\theta}{1 - [\Phi(r)]^2}$$
$$\leqq 2\pi[\rho(R_n, r') - \rho(R_n, r)]/[(1 - R_n^2)\rho(R_n, R_{n+1})]$$
$$\leqq 2\pi\,\rho(r, r') \leqq 2\pi\,\rho(z, z').$$

From the fact that $\rho(\Phi(r), \Phi(r')) \leqq \rho(r, r')$, we obtain

$$\rho(\Phi(r)\exp(i\beta + i\Psi(r')), {}_{\phi(r')}\exp(i\beta + i_{\Psi}(r')))$$
$$= \rho(\Phi(r), \Phi(r') \leqq \rho(r, r') \leqq \rho(z, z').$$

Combining the above estimates, we choose $K = 2 + 2\pi$ and the proof of the lemma is complete.

Proof of the Theorem. Let h be the homeomorphism of the lemma. Let g be a non-constant normal meromorphic function in D. Then by Theorem A, the light interior function $f = g \circ h$ is normal. If f has a point asymptotic value c along a Jordan arc Γ, then it is easy to verify that $h(\Gamma)$ is a spiral asymptotic path of g for the value c. By a theorem of Bagemihl and Seidel [1, Theorem 1, p. 10], $g \equiv c$ in violation of our hypothesis. Therefore f has no point asymptotic values and the theorem is proved.

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